AXIOMATISATION OF THE PROPOSITIONAL CALCULUS IN C AND Π.[1]

A.N. Prior and R.P. Kerr

By ‘the propositional calculus in C and Π’ is meant that form of propositional calculus in which the only undefined symbols are the propositional variables ‘p’, ‘q’, ‘r’, etc., the universal quantifier ‘Π′ and the symbol of implication ‘C’. This has been called by Russell, in (1),[1] the ‘theory of implication.’ In Łukasiewicz and Tarski’s (2)[2] it is stated that any set of axioms which, with the rules of substitution and detatchment, is sufficient for that segment of the propositional calculus which employs no operator but ‘C’ (e.g. the Tarski-Bernays axioms CCpqCCqrCpr, CCCpqpp and CpCqp) will become sufficient for the full calculus in C and Π if we add the following two rules of inference (‘x’ being a variable of any kind)

RT1. From Cαβ, we may infer CαΠxβ, provided that x does not occur freely in α.
RT2. From CαΠxβ, we may infer Cαβ.

This result is credited to Tarski, but we have been unable to find any published proof of it. In what follows, we shall show that Tarski’s result can be proved if a certain other result can be proved, and we shall sketch the proof of this other result.

In a number of comparatively recent writings of Łukasiewicz, e.g. (3), the following two rules for the introduction of the universal quantifier –{2}– have been employed: –

Π1. From Cαβ, we may infer CΠxαβ
Π2. From Cαβ, we may infer CαΠxβ,
provided that ‘x’ does not occur freely in α.

Π2 and RT1 above are identical; Π1 is not identical with RT2. But if Π1, with substitution and detachment, is employed in any system containing the theses Cpp and CCpqCCqrCpr, RT2 may be obtained as a derivative rule; for Cαβ will always be obtainable from CαΠxβ as follows: –

1. CαΠxβ
2. Cpp
3. CCpqCCqrCpr
   2 p/β X Π1 = 4.
4. CΠxββ
   3 p/α, q/Πxβ, r/β = C1 – C4 – 5
5. Cαβ

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[1] This text has been edited by Max Cresswell and Adriane Rini. The original is kept in the Prior collection, box 5 at the Bodleian Library, Oxford. [Editors’ note: footnotes 1 and 2, indicated by superscripts in the text are amalgamated at the foot of page 1. The text at the foot of that page reads: “1, 2. For our information as to the contents of these two papers, we are indebted to Professor A. Church.”

and conversely, if RT2, with substitution and detachment, is employed in any system containing the same two theses, Π1 may be obtained as a derivative rule, for CΠxαβ will always be obtainable from Cαβ as follows: –

1. Cαβ
2. Cpp
3. CCpqCCqrCpr
    2 p/Πxα = 4
4. CΠxαΠxα
    4 X RT2 = 5 –{3}–
5. CΠxαα
    3 p/Πxα, q/α, r/β= C5 – C1 – 6
6. CΠxαβ.

Hence if the rules Π1 and Π2 will give a sufficient basis for the calculus in C and Π if adjoined to the rules of substitution and detachment, with axioms sufficient for the calculus in C, the same will be true of the rules RT1 and RT2. (For Cpp and CCpqCCqrCpr are valid formulae of the calculus in C).

Rule Π1 above, adjoined to the purely implicational calculus will suffice to prove all valid formulae in C and Π in which Π occurs only as forming the proposition Πpp. For (i) if Π1 be applied to the implicational thesis Cpp, we obtain CΠppp; (ii) if '0' be used as an abbreviation for 'Πpp', this law CΠppp will become C0p; and (iii) Wajsberg has shown in (4) that if we introduce into propositional calculus a constant false proposition 0, a sufficient basis for the calculus in C and 0 may be obtained by adjoining C0p to any set of axioms sufficient for the calculus in C.

There are, however, other ways in which Π may occur in theses of the C – Π calculus: e.g. the way in which it occurs in the thesis CΠpCpqq. If, however, within the C – Π system formed by adjoining the rules Π1 and Π2 to the calculus in C, it can be shown that all formulae in which Π occurs in any way are deductively equivalent to formulae in which it occurs only in forming the proposition Πpp, this system can be considered complete. We do not propose here to prove this deductive equivalence, but we shall indicate where the proof can in effect be found. –{4}–

In (5), C.A. Meredith develops a system which uses not only 'C', 'Π' and propositional variables, but also the 'functorial' variable 'δ', so used that δα may represent any truth-function of which α is an argument. He shows that where '0' is an abbreviation for 'Πpp', 'Nα' for 'Cα0', '1' for 'N0' and 'Kαβ' for 'NCαNβ', the following theses are provable by substitution and detachment from the single axiom CδδΠppδp: –

8. Cpp
9. CCpqCCprCpKqr
10. CKδ0δ1δp.

He then (p. 44) uses these, with Π1 and Π2, to show that for any truth-function δq, Πqδq implies and is implied by Kδ0δ1. In the present paper we are concerned not with a C-δ-Π system but a C-Π one; but Meredith's proof will do for this also, if transposed a little. Of his premises 1-3, the first two contain no functorial variables, and so are provable in Wajsberg's C-0 calculus (with 'Nα' defined as 'Cα0' and 'Kαβ' as 'NCαNβ', and consequently in that part of the C-Π calculus with may be regarded as an interpretation of Wajsberg's. 3, CKδ0δ1δp, though it cannot be treated as an actual thesis within Wajsberg's calculus, can be treated as a metalogical abbreviation for a whole class of C-0 theses. Thus regarded, it asserts that within Wajsberg's calculus we may prove any formula which consists of a 'C', followed by a 'K', followed by any truth-function of '0', followed by the same truth-function of '1', followed by the same truth-function of 'p'. And in this sense it is proved in Quine's (6); or more accurately, Quine there proves the {5} metalogical counterpart of Cδ0Cδ1δp, from which that of CKδ0δ1δp follows easily by importation. Meredith's proof of CΠqδqKδ0δ1 and CKδ0δ1Πqδq may then be subjected to a similar re-interpretation.

It remains to show that in the C – Π system, if for any α and β – e.g. any formulae of the forms Πqδq and Kδ0δ1 – we have both Cαβ and Cβα as theses, we may replace either by the other in any formula and obtain a formula which implies and is implied by the original one. In the C—Π system the only ways in which a formula α may be built into a larger formula are (i) by its being the antecedent of an implication Cαγ or the consequent of an implication Cγα, or (ii) by its being preceded by a universal quantifier, i.e. by its occurring in a formula Πxα, where 'x' is any propositional variable; or (iii) by some succession of these procedures. From any set of axioms sufficient for the calculus in C we may derive the theses CCpqCCqrCpr and CCqrCCpqCpr; and by means of these, and Π1 and Π2, we can always derive from the theses Cαβ and Cβα the further theses CCαγCβγ, CCβγCαγ, CCγαC γβ, CCγβCγα, CΠxαΠxβ and CΠxβΠxα, as follows: –

1. Cαβ
2. Cβα
3. CCpqCCqrCpr
4. CCqrCCpqCpr
   3 p/β, q/α, r/γ = C2 – 5.
5. CCαγCβγ
   3 p/α, q/β, r/γ = C1 – 6. –{6}–
6. CCβγCαγ
   4 p/γ, q/α, r/β = C1 – 7
7. CCγαCγβ
   4 p/γ, q/β, r/α = C2 – 8
8. CCγβCγα
   1 X Π1 X Π2 = 9
9. CΠxαΠxβ
   2 X Π1 X Π2 = 10
10. CΠxβΠxα

Canterbury University College, Christchurch, New Zealand.

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