Letter from Saul Kripke to A.N. Prior, September 3, 1958[1]

119 North Happy Hollow Blvd.
Omaha 32, Nebraska
September 3, 1958

Professor A.N. Prior
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I have been reading your book Time and Modality with considerable interest. The interpretations and discussions of modality contained in your lectures are indeed very fruitful and interesting. There is, however an error in the book which ought to be pointed out, if you have not learned of it already. Although I know of your paper on "Diodoran Modalities" only from the references in your book and from the review in The Journal of Symbolic Logic, it seems that the correction applies to this article as well. (I would greatly appreciate it if you could send me a copy of this article.)

The infinite matrix given for S4 (p. 23) verifies the formula ALMpLMNp. For if the sequence for p does not end with an infinite series of 3's, the sequence for LMp will contain nothing but 1's; and if the sequence for p does end with an infinite series of 3's, Np will not end with such a sequence, and hence LMNp will be all 1's. However, McKinsey and Tarski (J.S.L. XIII 1) have shown that ALαLβ cannot be an S4 thesis unless either Lα or Lβ is an S4 thesis. Hence ALMpLMNp is not a S4 thesis, and thus your matrix is not characteristic of S4. It seems to me, however, that it probably ought to be an infinite characteristic matrix for a system intermediate between S4 and S5. The axioms of the system, over and above the two valued base, are (with L primitive): CLpp, CLpLLp, CLCpqCLpLq, and CKMpMqAMKpMqMKqMp. The last axiom is not valid in S4, but it is verified by your infinite matrix. The rules of inference are detachment, substitution, and from a thesis α to infer Lα. I propose to call the resulting modal system "the Diodoran system", since it is presumably the system Diodorus would have favored.

Your infinite matrix for T (p. 24) suffers from similar difficulties; it also satisfies ALMpLMNp. In fact it contains the fiction that there are only two possible times, the present moment and the one immediately next; the moment after next is not possible, merely possibly possible. Thus the infinite matrix verifiesCKMKpqMKpNqLp. I think we could axiomatize the system by CLpp, CLCpqCLpLq, and CKMKpqMKpNqLp, using the same rules as before. The resulting system is stronger than T, but neither stronger nor weaker than S3, S4, S5, Q, and theDiodoran system. It contains an infinity of modalities.

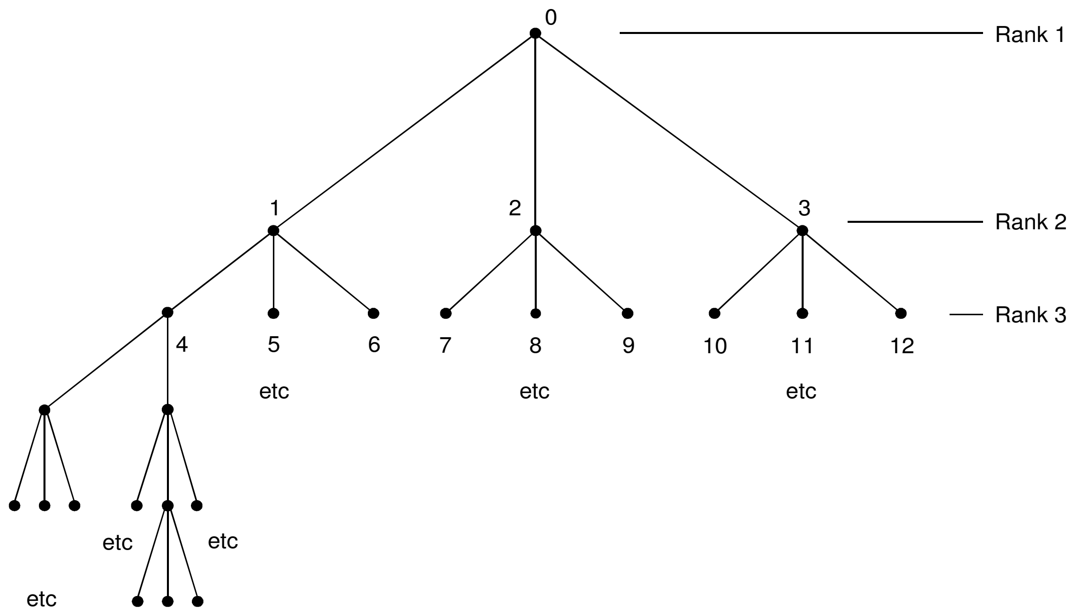
In view of my comments on S4, the "theorem" of footnote 1, p. 121, seems rather questionable, if conformity to S4 means not only containing all theorems of S4 but also lacking its non-theorems. Incidentally, your matrix for S5 seems indeed to be valid; in fact, it is in a sense a special case of a more general theorem of my own which will appear in J.S.L. sometime in the future. However, when I wrote the paper, I did not know about your fruitful interpretation in terms

[1] Edited by Thomas Ploug and Peter Øhrstrøm. The letter is kept at Bodleian Library, Oxford. - An earlier edition has been published in *Synthese* (2012) 188, p. 372 ff. The correspondence between Prior and Kripke has been discussed in Thomas Ploug & Peter Øhrstrøm: "Branching Time, Indeterminism and Tense Logic. Unveiling the Prior-Kripke letters", *Synthese* (2012) 188: 367-379.

of an infinitely many valued logic. It does seems to me to be unnecessary to allow a non-denumerable infinity of sequences as you have done; if we restrict the sequences to those ending in infinite series of l's or 3's, we seem to lose no generality—and the number of truth values becomes denumerable, indeed effectively enumerable.

If the notions of your book are suitably modified, an infinite charac-matrix for S4 can also be obtained, as I will indicate below.

I have in fact obtained this infinite matrix on the basis of my own investigations on semantical completeness theorems for quantified extensions of S4 (with or without the Barcan axiom). However, I shall present it here from the point of view of your "tensed" interpretation. (I myself was working with ordinary modal logic.) The matrix seems related to the "indeterminism" discussed in your last chapters, although it probably cannot be identified with it. Now in an indetermined system, we perhaps should not regard time as a linear series, as you have done. Given the present moment, there are several possibilities for what the next moment may be like—and for each possible next moment, there are several possibilities for the next moment after that. Thus the situation takes the form, not of a linear sequence, but of a "tree":[2]



The point 0 (or origin) is the present, and the points 1, 2, and 3 (of rank 2) are the possibilities for the next moment. If the point 1 actually does come to pass, 4, 5, and 6 are <u>its</u> possible successors, and so on. The whole tree then represents the entire set of possibilities for present and future; and every point determines a <u>subtree</u> consisting of its own present and future. Now if we let a tree sequence attach not three (as above) but a denumerable infinity of points to every point on the tree, we have a characteristic matrix for S4. An element of the matrix is a tree, with either 1 or 3 occupying each point; the designated tree contains only l's. If all points on the proper "subtree" determined by a point on the tree p are 1's, the corresponding point on Lp is a 1; otherwise, it is a 3. (In other words, a proposition is considered "necessary" if and only if it is and definitely always will be the case.) Analogously, for T, we can change the stipulation so that a point on Lp is a 1 if the corresponding point on p, and all the points immediately attached to it (i.e., possibilities for

[2] The illustration has been remade on the basis of Kripke's drawing.

moments immediately following it), are l's; otherwise, it is a 3. As I have described the situation, the matrices for S4 and T are non-denumerable; but certain restrictions which I shall not describe here enable us to obtain a denumerably valued matrix for each system. (Because of the correspondence between S4 and the Heyting systerm, a denumerable matrix for the latter is also obtained, which should be compared to Jaskowski's. I have not yet carried out this comparison. McKinsey–Tarski (J.S.L. XIII l) also mention infinite matrices for these systems—S4 and Heyting's.)

Your system Q represents an ingenious piece of analysis, and I intend to investigate it by my own semantical methods. I think I should probably be able to obtain an axiomatization as a by-product. When I work such matters out, I shall write to you about them.

My corrections to your matrices for S4 and T may already be known to you. If they are not, I think a correction should be published somewhere, perhaps when J.S.L. reviews your book. If you know who the reviewer will be, you could write to him in my name or have me write to him. If not, you could write to Church, or try any other alternative you may desire. If you can reply to this letter to arrive before September 10th, write to the given return address; otherwise write to Harvard College, Weld House, Cambridge 38, Massachusetts, USA.

Yours very sincerely,
Saul A. Kripke[3]

[3] Added on the letter in Prior's hand: "Passed on to Ivo and John L."