Postulate Sets for Tense Logic[1]

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(All to be subjoined to p.c. with substitution and detachment)

I. Without Specific Intervals: The System GH

Intuitively,

Fp = it will be the case that p
Gp = it will always be the case that p
Pp = it has been the case that p
Hp = it has always been the case that p.

Subjoin to p.c. with substitution and detachment the following:

Definition: F = NGN; P = NHN (G & H primitive)
Rules:

RG: ├ α → ├ Gα

MI: (Mirror Image Rule): We may simultaneously replace F by P, G by H, and vice versa, throughout a thesis, and the result will be a thesis.

Axioms:

1. CGCpqCGpGq
2. CGpNGNp (= CGpFp)
3. CG*p*GG*p*
4. CGG*p*G*p*
5. CGCpqCGCpGqCGCFppCFpGq
   (= CKFpFqAAFKpqFKpFqFKFpq)
6. CNHNGpp (= CPGpp)
7. CNHNGpGp (= CPGpGp)
8. CGpGNHNGp (= CGpGPGp)
9. CpCH*p*CG*p*HG*p*

Query: Do Df.F, Axx. 1-5 and RG suffice for the pure FG fragment?
With Df. L: Lp = KpGP, the L fragment of GH is S4.3; with Lp = KKpGpHp, the L fragment is S5.

II. With Specific Intervals

Intuitively, Fn*p* = it will be the case the interval n hence.
Pn*p* = it was the case the interval n ago.

Variables n, m etc. stand for real numbers within a certain range, measuring a time-interval; and (with the usual provisos in the presence of quantifiers) we may substitute for any such variable

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throughout a thesis (a) another such variable, (b) a particular real number from within the range, or (c) a sum or difference of real numbers, provided this sum or difference falls within the range. At any place in a thesis we may also substitute for a given expression denoting a real number, any other denoting the same real number, e.g. (m + n – m) for n, or 0 for (m – m), or (m + n) for (n + m).

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II.1: The System P- (P minus)
Here m, n etc. stand for any real numbers, positive, negative and zero. P is primitive.

Definition: Fnα =P(–n)α

Rule: RP: ├ α → ├ Pnα
Π1: Cαβ → CΠnαB
Π2: Cαβ → CαΠnB, for n not free in α.
Axioms: P0: EP0pp
PN: EPnNpNPnp
PC: EPnCpqCPnpPnq
PP: EPmPnpP(m + n)p
PΠ: CΠnPmΦnPmΠnΦn
If in this system we define L as ΠnPn (or as ΠnFn), its L-fragment will be S5.

II.2: The System PF0
Here m, n etc. stand for real numbers ≥ 0. P and F are distinct primitives, so there is no Df.F.

Rules: RP, Π1, Π2 as before, plus:

Rmn: If a formula is a thesis under the proviso that m ≥ n, and also under the proviso that n ≥ m, it is a thesis simpliciter.
MI: From any thesis we may obtain another thesis by replacing P by F and F by P throughout.

Axioms: All those of the previous system, plus:
PF1: EPmFnpP(m – n)p, for m ≥ n
PF2: EPmFnpF(n – m)p, for n ≥ m

If L is defined in this system as ΠnFn (or ΠnPn), then the resulting L-fragment is S4.3.

II.3: The System PF
Here m, n etc. stand for real numbers > 0. P and F are again distinct primitives.

Rules and Axioms: are as in the preceding, except that;
(i) The axiom P0 (EP0pp) is dropped as meaningless;
(ii) The rule Rmn is re-stated thus:-
Rmn: If a formula is a thesis under each of the provisos that m > n, that n > m, and that m = n, then it is a thesis simpliciter.
iii) The axioms PF1 and PF2 are replaced by:

PF1: EPmFnpP(m – n)p, for m > n.
PF2: EPmFnpF(n – m)p, for n > m.
PF3: EPnFnpp;

{3} iv) The following axiom is added
PPΠ: CΠlΠmPlPmpPnp (here ‘l’ is the small case letter L).

If, in this system we define G as ΠnFn and H as ΠnPn, its GH fragment becomes the system GH of I. (PPΠ is needed to obtain CGGpGp, which presupposes the continuity of time).

III. PFL Systems

These systems abandon the symmetry between past and future which characterises the preceding ones, and attempt to embody the idea that it is not within our power to make some but not other propositions true or false. In particular, most past tensed propositions are beyond our power to make true or false – in the first system below, all past tensed propositions except those with an element of futurity about them; e.g. it may be still within our power to make true or false the proposition that it <u>was</u> the case yesterday that we would be smoking two days later (we make it true by deciding now to smoke tomorrow, false by deciding not to). For the detailed philosophical motivations of these systems, see the article ‘The Formalities of Omniscience’, <u>Philosophy</u>, April [1962], pp. 114-129.

III.1: The System *O* (or Occamist PFF System)

This system adds to a tense logical basis a primitive necessity-operator L, which indicates that it is now beyond out power to make true or false the proposition that follows it.

The system also contains beside the ordinary propositional variables *p*, *q*, *r* etc. (P-variables), which may stand for propositions of any sort, a set of special propositional variables *a*, *b*, *c*, etc. (A-variables), standing only for those propositions which it is now beyond our power to make true or false. (Hence ├ CaLa, but ┤CpLp). There are two corresponding sorts of well-formed formula, defined as follows:-

<u>P-formulae</u> (substitutable for P-variable):-

(1) All propositional variables (A or B) are P-formulae.
(2) If α and β are both P-formulae, so are Nα, Cαβ (Kαβ, etc.), Lα, Pnα and Fnα.
(3) There are no others.

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<u>A-formulae</u> (substitutable for A-variables):-

(1) All A-variables are A-formulae.
(2) If α and β are both A-formulae, so are Nα, Cαβ (Kαβ, etc.), and Pnα (<u>not</u> Fnα).
(3) If α is a P-formula, Lα is an A-formula.
(4) There are no others.

The purely tense-logical postulates are all those of the system PF0 of II.2, except for the mirror image rule. In its place we have the mirror images of the particular postulates RP, PN, PC, PP, PΠ, PF1 and PF2, which we may call RF, FN, FC, FF, FΠ, FP1 and FP2.

To these we add the following for L:-

<u>Rule</u>: <u>RL</u>: ├ α ⇒ ├Lα

<u>Axioms</u>:

<u>L1</u>. CLp
<u>L2</u>. CLCpqCLpLq

<u>L3</u>. CNLpLNLp
<u>LA</u>. CaLa
<u>FL</u>. CLFmFnpLFmLFnp

RL and L1-3 give us S5 for the primitive L. If we define a tense-logical L as ΠnFn, for this L we have S4.3, as in PF0.

III.2: The System P (or Peircean PFL)

This system consists of all those theses of the preceding system O in which:

(a) the symbol F does not occur except as immediately preceded by an L, and
(b) all variables are A-variables;

these theses being re-written as follows:-
(i) the A-variables are replaced by P-variables,
(ii) the symbol L is deleted throughout.

This system has not been axiomatised, but the following points about its axiomatics may be noted :-

(1) We may discover whether a formula is a thesis of P by finding out whether the same formula with all Fs replaced by LFs and all P-variables by A-variables, is a thesis of O. For example CpFnPnp is provable in P because CaLFnPna is provable in O, as follows:-

1.	EF0P0pP0p	(FP1; 0 – 0 = 0)
2.	EF0P0pF0p	(FP1; 0 – 0 = 0)
3.	EF0pP0p	(1, 2)
4.	EF0pp	(3, PO)
5.	CaF0a	(4)
6.	CaF(n – n)a	(5; n – n = 0)
7.	CaFnPna	(6; FP1)
8.	LCaFnPna	(7; RL)
9.	CLaLFnPna	(8; L2)
10.	CaLFnPna	(9; LA)

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On the other hand, CpPnFnp, which is the mirror image of CpFnPnp, is not provable in O. It is provable without the L, but we could only insert an L at that point if we had CFnaLFna which is not derivable from CaLa because Fna is not an A formula.

(2) This criterion leaves us, in P, with all of the following postulates of O: Π1, Π2, Rmn; RP, PO, PN, PC, PP, PΠ; also RF, FC, FF, F , FP1 and PF2. But FN, PF1 and PF2 must be weakened to the following implications:
FN(P): CFnNpNFnp
PF1(P): CPmFnpP(m – n)p, for m ≥ n
PF1(P): CPmFnpF(n – m)p, for n ≥ m

(3) The weakening of FN means that we need a new axiom FK: CKFnpFnqFnKpq, as this is no longer provable from FN and FN (and Df. K: Kpq = NCpNq) as in PFO and O.

If we define a tense logical L in the system P as ΠnFn, the L-fragment is no longer S4.3, but only S4. If we define M independently of L as ΣnFn (i.e. NΠnNFn) instead of as NLN, the LM-fragment is another modal system again, contained in S4.3 and independent of S4 (it has Hintikka's S4.3 axiom CKMpMqAMKpMqMKMpq, which is not in S4, but lacks such S4 theses as CNMpMNp).

III.3: The Systems O' and P'

The Occamist system O' is like O except in drawing its tense-logical basis from the system PF of II.3 instead of from the system PFO of II.2. It takes from PF all postulates but the mirror-image rule MI, using instead the mirror images of RP, PN, PC, PP, PΠ, PF1, PF2 and PF3. If within this system we define G as ΠnFn and H have, for our GH-fragment, as with PF, the system GH of I.

The system of P' is formed from O' in the same way as P from O, and similarly has to weaken FN, PF1, PF2 and PF3 to implications. With G = ΠnFn, and H = ΠnPn, its GH fragment is weaker than that of O', much as the tense-logical L-fragment of P (i.e. S4) is weaker than that of O (i.e. S4.3). We no longer have, for example, CFGpGFp, i.e. CNGNGpGNGNp, though in the system GH this follows easily from Axiom 5 (put Gp for p, Fp for q, and detach successive antecedents). We may also obtain a different GHPF system within P' by defining the monadic F as ΣnFn instead of NGN. With this definition we do have CFGpGFp, but do not have e.g., CpHFp (though we do have its mirror image CpGPp).

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III.4 Models for PFL Systems

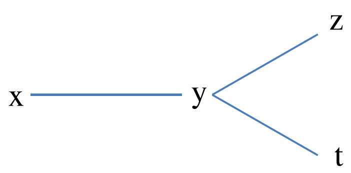
In these models the course of time (in a rather broad sense of this phrase) is represented by a line which, as it moves from left to right (past to future), continually divides into branches, so that from any given point in the diagram there is a unique route backwards (to the left; to the past) but a variety of routes forwards (to the right; to the future). In each model there is a single designated point, representing the actual present moment; and in an Occamist model there is a single designated line (taking one only of the possible forward routes at each fork), which might be picked out in red representing the actual course of events.

We begin with a model for that part of the system O in which all variables are A-variables. This includes, it may be noted, all substitution-instances involving only A-variables, of theses involving P-variables, many of which could not be obtained by substitution in theses involving only A-variables. For example, it includes CLFnaFna, which is a substitution instance of CLpp but not (since Fna is not an A-formula) of CLaa. In a model for this system, truth-values may be assigned to all its formulae at every point in the model-diagram, in accordance with the directions below. A distinction is made between the actual assignment of values at each point, and various prima facie assignments.

(1) Each A-variable is arbitrarily assigned an actual truth-value at each point, and this is its only prima facie assignment at that point.
(2) A prima facie assignment to FnΦ at a point x will give it the value assigned to Φ at the distance n along some path to the right of x (where the diagram forks within this distance, FnΦ will have a number of different prima facie assignments at x).
(3) The actual assignment to FnΦ at x gives it the value of α at the distance n to the right along the designated line.

(4) An actual or prima facie assignment to Pnα at x will give it the actual or prima facie assignment given to α at the distance n to the left of x.
(5) The actual assignment to LΦ at x will give it truth if all the prima facie assignments to Φ at x give it truth; otherwise falsehood. This is the only prima facie assignment to LΦ at x.
(6) Truth functions and quantifications are assigned their values in the usual way.

Consider, for example, the following simple model:



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where the distance xy is n and the distance yz and yt are both m; the designated path is xyz; and the proposition a is true at x, y and z and false at t. F(n + m)a is actually assigned truth at x. But one prima facie assignment, namely that in which our movement to the right is along xyt, would give falsehood to F(n + m)a at y. This in turn means that we must actually assign falsehood to LPnF(n + m)a at y, and therefore (since PnF(n + m)a is actually assigned truth at y) falsehood at y to:

CPnF(n + m)a LPnF(n + m)a

This formula is therefore falsifiable in an O-model, and thus is not a theorem of this portion of O; and thus CPnpLPnp, of which it would be a substitution-instance, is not a theorem of O itself. In O we have some "power over the past".

In a model for the system P, we have no designated line and no functor L, but assign values as follows:-
(1) All variables are arbitrarily assigned truth-values at all points.
(2) A prima facie assignment to Fnα at x is as in the model for O.
(3) The actual assignment to Fnα at x gives it truth if all its prima facie assignments are given truth; otherwise falsehood.
(4) The actual assignment to Pnα at x gives it the value assigned to α at the distance n to the left of x; and this is its only prima facie assignment.
(5) Truth functions and quantifications as usual.

In the simple model given above, F(n + m)a is false at x, and PnF(n + m)a therefore false at y. Also, although a is true at z, Fma is false at y, and PmFma therefore false at z, so that CaPmFma is falsifiable in a P-model, and is not a law of P. Neither, therefore, it the equivalent CP(m–n)aPnFma, or the principle of which this would be a special case, i.e.P(m – n)PmFna (m ≥ n), the converse of PF1 (P). Past and future (taken in that order) may be compounded but not divided in P.

Addendum to II

Results of Rescher and Meyer make it clear that the following modifications of PF-,PFO, PF, O, and O' will not weaken the systems: Change E to C in PO, PP, PF1, PF2, PF3, and their mirror images. But except in the case of the PFs, the originals must be used in P and P'.