UNIVERSITY OF CALIFORNIA, LOS ANGELES

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Dear Professor Prior,

In your 'Egocentric Logic' [Noûs Aug 68], you appear to wonder whether the quantificational definition of *Q*p has any quantifier-free consequences not derivable from the quantifier-free axiom schemas Q1-Q9. It does not.

If we take $\langle w\ \mathrm{W}\ \mathrm{F}\rangle$ to be a model in the now familiar Kripke way (W a set of poss. worlds, $w \in \mathrm{W}$, if P is a sent. constant, $\mathrm{F}(P) \subseteq \mathrm{W}$) and give the natural definitions of truth in $\langle w\ \mathrm{W}\ \mathrm{F}\rangle$ with the special clause:

$Q\Phi$ is true in $\langle w\ \mathrm{W}\ \mathrm{F}\rangle$ iff there is exactly one $w' \in \mathrm{W}$ s.t. Φ is true in $\langle w'\ \mathrm{W}\ \mathrm{F}\rangle$

then, it can be shown that your Q1-Q4 (plus the usual S5 stuff) are a complete (and consistent) set. Since the definition of *Q* is also valid (with the natural clause for the propositional quantifiers), any quant-free consequence of that definition is also valid and hence already derivable from Q1-Q4

The completeness proof goes by way of normal forms (there may be an even easier argument some other way). Let $P_1 \ldots P_m$ be distinct sentence letters. Then there are exactly 2^m *state-descriptions* of the form $\in_1 P_1 \wedge \ldots \wedge \in_m P_m$ where each $\in_i$ is either a blank (or alternatively $\neg\neg$) or the negation sign. Let us call the state-descriptions: $S_1 \ldots S_{2^m}$. A *reduced model-description* (in $P_1 \ldots P_m$) is a conjunction of the form $\Gamma \wedge \alpha_1\Diamond S_1 \wedge \ldots \wedge \alpha_{2^m}\Diamond S_{2^m} \wedge \beta_1\Diamond S_1 \wedge \ldots \wedge \beta_{2^m}\Diamond S_{2^m}$ where (i) $\Gamma = S_j$ for some $1 \leq j \leq 2^m$, (ii) each α_j and each β_j is either a blank or the negation sign (for all $1 \leq j \leq s^m$), (iii) if β_j is a blank then so is α_j (for all $1 \leq j \leq 2^m$), and (iv) α_j is a blank for the j ($1 \leq j \leq 2^m$) such that $\Gamma = S_j$.

LEMMA 1: If D is a reduced model-description (in $P_1 \ldots P_m$), then there is a model in which D is true and all other reduced model-descriptions in $P_1 \ldots P_m$ are false.

Proof: Let the poss. worlds be of the form $\langle 0,S_j\rangle$ or $\langle 1,S_j\rangle$ for $1 \leq j \leq 2^m$
$\langle 0,S_j\rangle \in \mathrm{W}$ iff α_j is a blank
$\langle 1,S_j\rangle \in \mathrm{W}$ iff α_j is a blank and β_j is the negation sign
$w = \langle 0,\Gamma\rangle$
If P is a sentence letter, and $x = 0$ or $x = 1$, then $\langle x,S_j\rangle \in \mathrm{F}(P)$ iff $S_j \vdash P$

An easy induction completes the proof.

LEMMA 2: If Φ is any quantifier-free formula, there exists a formula χ such that [χ is a disjunction of reduced model-descriptions *or* χ is $\bot$ (a convenient sentential contradiction)] and χ is provably equivalent to Φ.

Proof: First put Φ into a provably equivalent form in which no modal operator (including Q) stands within the scope of another. That this can be done in S5 is a familiar fact. Here we use especially three principles: $\vdash (QP \leftrightarrow \Box QP)$ [this is used to confine across a Q], $\vdash [Q(p \vee q) \leftrightarrow \{(qp \wedge \Box(p \leftrightarrow q)) \vee (Qp \wedge \Box\neg q) \vee (Qq \leftrightarrow \Box\neg p))\}]$ [This is used to get the operand of q to be a conjunction by putting the original operand into disjunctive normal form and then using the principle to distribute], $\vdash (Q(\Box p \wedge r) \leftrightarrow (\Box p \wedge Qr))$ [this is used to confine Q to the part of its operand (when a conjunction) which contains no modal operators]. To see how to reduce the "degree" of an operator consider a formula OpΓ where Op is either $\Box$ or Q, if Op is $\Box$ we put Γ in conjunctive normal form and distribute, the disjuncts can be put in the form A $\vee$ B where A is modally closed (every sent. let. in the scope of an operator) and B is modally open-free. Using the first of the three principles we easily prove by induction that if any formula Δ is modally closed; $\vdash (\Delta \leftrightarrow \Box\Delta) \wedge (\neg\Delta \leftrightarrow \Box\neg\Delta)$. Thus $\vdash (A \leftrightarrow \Box A)$ and hence $\vdash [\Box(A \vee B) \leftrightarrow (A \vee \Box B)]$. This will give a formula provably equivalent to OpΓ and of lower degree (unless OpΓ already had no overlay[*]). If Op is Q put Γ into disjunctive normal form and distribute by principle two. Components of the form A $\wedge$ B with A modally closed and B modal-free. Again using the fact that $\vdash A \leftrightarrow \Box A$ and the third principle we obtain $\vdash (Q(A \wedge B) \leftrightarrow A \wedge QB)$ which gives s formula of lower degree (unless OpΓ had no overlay). Here are sketches of the proofs for the second and third principles.

~~show~~ $Q(p \vee q) \leftrightarrow (Qp \wedge \Box(p \vee q)) \vee (Qp \wedge \Box\neg q) \vee (Qq \wedge \Box\neg p)$

From left to right: case 1: Ass $Qp \wedge \Box(p \leftrightarrow q)/\Box(p \leftrightarrow p \vee q/$ by Q4 $Q(p \vee q)$
case 2: Ass $Qp \wedge \Box\neg q)/\Box(p \leftrightarrow p \vee q/$ "
case 3: As in case 2

[*] [Editor's note] I'm uncertain of this word.

From right to left: Ass $Q(p \vee q)$, the following cases are by Q2

case 1: $\Box(p \vee q \rightarrow p \wedge q / \Box(p \leftrightarrow p \vee p$/by Q4 Qp,
Also by hyp of the case $\Box(p \leftrightarrow q)$
case 2: $\Box(p \vee q \rightarrow \neg(p \wedge q)/(1)\ \Box(p \rightarrow \neg q, \Box(q \rightarrow \neg p)$
By the Ass and Q1/$\Diamond(p \vee q)/\Diamond p \vee \Diamond q$ which gives the foll. subcases
Subcase a: $\Diamond p$

~~show~~ Qp By the Ass and Q2 we have 2 more cases

case a_1: $\Box(p \vee q \rightarrow p)/\Box(p \vee q \leftrightarrow p)$/by Q4 Qp
case a_2: $\Box(p \vee q \rightarrow \neg p)/\Box\neg p/\neg\Diamond p$ but this contradicts the hypothesis of the subcase

~~show~~ $\Box\neg q$ By the Ass and Q2 we have 2 more cases

case a_3: $\Box(p \vee q \rightarrow q)$/using (1) $\Box(p \rightarrow \neg p)/\neg\Diamond p$ which contradicts hyp. of the subcase
case a_4: $\Box(p \vee q \rightarrow \neg q)/\Box(q \rightarrow \neg q)/\Box\neg\Diamond q$

Subcase b: $\Diamond q$ as for subcase a

~~show~~ $Q(\Box p \wedge r) \leftrightarrow (\Box p \wedge Qr)$

From right to left, Ass Qr, $\Box p/\Box((\Box p \wedge r) \leftrightarrow r)$/by Q4 $Q(\Box p \wedge r)$
From left to right, Ass $Q(\Box p \wedge r)$/ by Q1 $\Diamond(\Box p \wedge r)/\Box p/\Box((\Box p \wedge r) \leftrightarrow r)$
by the Ass and Q4 $Qr/\Box p \wedge Qr$

We have been putting Φ into a provably equivalent form Φ' "without overlay". Now take each subformula of Φ which has either the form $\Diamond\Delta$ or $Q\Delta$, put Δ into *standard* disjunctive normal form (each disjunct a state-desc. in the sent. let's of Φ) or replace Δ by $\perp$ (if Δ is a sent. contradiction). Then distribute $\Diamond$ in the usual way and distribute Q by principle two. One iteration may be required since principle two produces some new formulas of the form $\Diamond\Delta$ (I take $\Diamond$ as an abb. for $\neg\Box\neg$, a little $\vdash p \leftrightarrow \neg\neg p$ may be required along the way). We now have a formula Φ'' provably equivalent to Φ' whose atoms are: the sent. let's of Φ, $\Diamond S_j$ (where S_j is a state desc. in the sent. let's of Φ), QS_j (S_0 as before), $\Diamond\perp$, $Q\perp$, $\perp$. Put Φ'' into a provably equivalent disjunctive normal form in these atoms. If a disjunct of this complete formula contains $\perp$, $\Diamond\perp$ or $Q\perp$, replace the disjunction by $\perp$. Now omit any disjunct, $\neg\perp$, $\neg\Diamond\perp$, and $\neg Q\perp$ and re-arrange remaining non-$\perp$ disjuncts in the form of a reduced model-description. If either condition (iii) or (iv) of a reduced model-description is not satisfied by any disjunct, replace that disjunct by $\perp$. If what remains contains a non-$\perp$ disjunct, drop all $\perp$'s, otherwise replace the whole formula with $\perp$. Each of the foregoing transformations is based on a provable equivalence. The final formula is the χ called for in Lemma 2.

LEMMA 3: If χ is a disjunction containing each of the reduced model-descriptions (in some fixed set of let's) as a disjunct, then χ is provable.

Proof: χ is provably equivalent to the tautology formed by adding additional disjuncts in the same atoms. these added disjuncts will all have the form of reduced model-descriptions but will violate either (iii) or (iv) and thus be provably false.

THEOREM: If χ is valid (and quantifier-free), χ is provable.

Proof: By lemma 2, χ is provably equivalent to a disjunction of red. model-desc's, which by lemma 1 and the validity of χ must contain all red. model-desc's (in the sent. let's. of χ, and which is thus by lemma 3 provable.

[I have assumed, what is easy, that all theorems are valid.]

The method of proof also, of course, yields decidability for the quant.-free formulas.

On the last few pages, you claim that certain formulas (e.g. Q6) do not follow from what amounts to the definition of *Q* in what you call "quantified S5". Since there is a complete [and decidable] axiomatization for this theory [S5 with prop. quant's] (complete in the sense of yielding all [and only] those theorems that are true in every model: with the clause

> $\forall P\Phi$ is satisfied by the assignment *f* in $\langle w\ W\ F\rangle$ iff for all $\check{P} \subseteq W$, Φ is satisfied by $f^{P}_{\check{P}}$ in $\langle w\ W\ F\rangle$.[**])

and since the definition of *Q* does define *Q* in such a way that $Q\Phi$ is true when exactly the conditions on p. 1 of this letter [p. 1 also of the transcription (Ed.)]), all valid formulas must be theorems. Since the formulas you claim underivable are clearly valid, I must conclude that you have in mind a defective axiomatization for quantified S5. My own axiomatization is as follows:

1. $\vdash \forall P[\Phi \rightarrow \Psi] \rightarrow (\forall P\Phi \rightarrow \forall P\Psi]$
2. $\vdash \forall P\Phi \rightarrow \Psi$ where Ψ is the result of replacing all free occ's of P in Φ by free occ's of some formula Γ.

[**][Editor's note] Kaplan has not in this letter introduced the notion of satisfaction with respect to an assignment, but it has to be that *f* is a function which assigns a set of worlds, which he refers to as $\check{P}$, to every propositional variable. $f^{P}_{\check{P}}$ then refers to the satisfaction of Φ by an assignment just like *f* except that it assigns to P the set of worlds $\check{P}$.

3. $\vdash \Phi \to \forall P\Phi$ where P is not free in Φ
4. $\vdash \Phi$ where Φ is a tautology
5. $\vdash \Box[\Phi \to \Psi] \to (\Box\Phi \to \Box\Psi]$
6. $\vdash \Box\Phi \to \Phi$
7. $\vdash \Phi \to \Box\Phi$ where Φ is modally closed
8. $\vdash \exists P[P \wedge \forall Q(Q \to \Box(P \to Q))$

Rules: Modus Ponens, Universal Generalization, Modal Generalization

My completeness proof goes by way of normal forms and is thus quite awful. I hope there is an easier proof.

Probably the system that you had in mind lacked Axiom 8, Kripke also seems top have missed it at the end of his March '59 J.S.L. article (so that the added remark on the last page is not strictly correct).

Possibly what you meant at bot. p. 206-top p. 207 was that there is no formula Φ, for which either $\Phi \wedge Q\Phi$, or even just $Q\Phi$ is provable. This is true and is most easily shown by model-theoretic methods.

The fact that quant. S5 is decidable (and this that there exists a complete axiomatization) [oops! I've misjudged the length of this letter***] follows from the fact that quant. S5 is isomorphic to a subtheory of 2^{nd} order monadic predicate logic, which is known to be decidable. When P is a sent. constant (or var.) of quant. S5, let P' be a uniquely determined monadic predicate constant (or var.). Now we define a translation of formulas of quant. S5 into those of 2^{nd} ord. monadic logic.

Trans(P) = $P'x$ (x: a chosen variable)
Trans($\Phi \wedge \Psi$) = (Trans(Φ) $\wedge$ Trans (Ψ))
Trans($\neg\Phi$) = $\neg$ Trans(Φ)
Trans($\forall P\Phi$) = $\forall P'$Trans(Φ)
Trans($\Box\Phi$) = $\forall x$Trans(Φ)

It is easy to show that an assignment (of prop's to sent. var's) f sat's in quant S5 Φ in $\langle w \text{ W F}\rangle$ iff fv$\{\langle x\ w\rangle\}$ sat's in 2^{nd} order logic Trans(Φ) in $\langle$W F$\rangle$. What is more difficult for me was establishing that the *intrinsic* axiomatization given above works.

Since your Q, W and O are interdefinable and the quantifier-free axioms you give are equivalent on the basis of the def's, what I have said for Q goes also for W and O.

***[editorial comment] The sentence in brackets in the text appears in red at the top of p. 7 of the ms.

Unfortunately my Polish is not very fluent so I have resorted to Lewis (?).

I am sorry not to have had the opportunity of meeting you when you were here, my colleagues were all most enthusiastic about your visit. I hope you will come to California again soon.

Sincerely

David Kaplan

PS Could you put me on your reprint list?